The heat absorbed by the fluid rising in the tube can be written as

$$Q = 2\int_0^1 uTr\,\mathrm{d}r. \tag{16}$$

RESULTS AND DISCUSSION

In order to justify the validity of the present linearized method, the results were compared with available exact numerical solutions to the problem. Figure 1 shows the calculated inlet velocity u_i vs Gr for $Pr = 0.7$ as well as the values given by Davis and Perona [2]. The numerical results of Kageyama *et al.* [1] for $Pr = 0.72$ is very much similar to that of [2], and are not shown in the figure. The numerical solutions correspond to zero inlet pressure, while the actual pressure must be lower since the fluid has been accelerated from rest [3, 4]. The good agreement shown between the results of the linearized version of the boundary layer equations with the exact numerical solutions verifies the propriety of the present method of analysis.

Representative temperature and velocity profiles at four levels of x are shown in Fig. 2, while the heat flux and pressure levels as a function of axial position are shown in Fig. 3. The pressure defect is decreased by large buoyancy forces for large x.

It must be noted, however, that the use of boundary layer type of equations in the analysis is limited to cases where Gr is not too large. Otherwise, the velocity profiles will indicate a downward flow at the center of the tube exit, and boundary-layer-type equations used would not be valid for this flow behavior. Finally, the Oseen type of linearization of the governing equations should only relate to physical cases where the length to diameter ratios of tubes considered are moderately large.

REFERENCES

1. M. Kageyama and R. Izumi, Natural convection in a vertical circular tube, *Bull. J.S.M.E.* **13**, 382–394 (1970).
2. L. P. Davis and J. J. Perona, Development of free convection flow of a gas in a heated vertical open tube, *Int. J. Heat Mass Transfer* **14**, 889–903 (1971).
3. J. Quintiere and W. K. Mueller, An analysis of laminar free and forced convection between finite vertical parallel plates, *J. Heat Transfer* **95**, 53–59 (1973).
4. T. Aihara, The effect of the entrance boundary conditions on free convection between vertical parallel plates, *Proceedings of the* 9th *National Heat Transfer Symposium* (*Japan*), p. 117. Hiroshima (1972).
5. E. M. Sparrow, S. H. Lin and T. S. Lungren, Flow development in the hydrodynamic entrance region of tubes and ducts, *Physics Fluids* **7**, 338–347 (1964).

Int. J. Heat Mass Transfer. Vol. 20, pp. 431–433. Pergamon Press 1977. Printed in Great Britain

CONVECTIVE HEAT TRANSFER TO LAMINAR FLOW OVER A PLATE OF FINITE THICKNESS

P. PAYVAR
Arya-Mehr University of Technology, P.O. Box 3406
Tehran, Iran

(*Received* 17 *June* 1976)

NOMENCLATURE

- b, plate thickness;
- Br_x, local Brun number defined by equation (1);
- K, thermal conductivity;
- Pr, Prandtl number;
- q, local heat flux;
- Q, transformed heat flux defined by equation (14);
- Re_x, local Reynolds number;
- T, temperature;
- u, v, longitudinal and transverse components of velocity in boundary layer;
- x, y, Cartesian coordinates;
- X, transformed coordinate defined by equation (18);
- Z transformed coordinate defined by equation (10);

Greek symbols

- α, thermal diffusivity;
- β, parameter defined by equation (17);
- δ, boundary-layer thickness;
- ξ, transformed distance defined by equation (9);
- τ, shear stress;
- μ, absolute viscosity;
- ν, kinematic viscosity;
- θ, dimensionless temperature defined by equation (6);
- Φ, dimensionless temperature, $\Phi = 1-\theta$.

Subscripts

- ω, refers to wall surface in contact with fluid;
- b, refers to wall surface at constant temperature;
- f, refers to fluid;
- s, refers to solid;
- T, refers to thermal boundary layer;
- x, refers to local values;
- ∞, refers to mainstream flow;
- 0, refers to values of Nu_x at $Br_x = 0$.

INTRODUCTION

IN THE usual formulation of the problem of heat transfer to flow over a flat plate, boundary conditions are specified at the upper surface of the plate which is in contact with the fluid. If, however, the boundary conditions are specified over the lower surface of the plate, the effect of plate resistance, if signigicant, must be included in the analysis resulting in a conjugate heat-transfer problem. This represents a more realistic approach and analyses of this type have recently received increased attention resulting in publication of a number of papers [1]. A formulation of such problems was originally presented by Luikov [2] and analytical methods of solution of certain conjugate problems were given by Luikov, Aleksashenko and Aleksashenko [3]. More recently Luikov [4] presented a solution of the problem of heat transfer to laminar flow over a plate of finite thickness with the lower surface of the plate maintained at a uniform constant temperature. His results indicate that for values of the Brun number, Br_x, greater than 0.1, neglecting the plate resistance will result in errors of more than 5%. The Brun number is defined as,

$$Br_x = \frac{K_f}{K_s}\frac{b}{x} Pr^m Re_x^n \tag{1}$$

and is a measure of the ratio of the thermal resistance of the plate to that of the boundary layer. Lighthill [5] has presented a method of analysis of laminar boundary-layer heat-transfer problems which yields reasonably accurate results at Prandtl numbers near unity and becomes more and more accurate as Pr increases. In the present study Lighthill's method is used to solve the problem of heat transfer to laminar incompressible flow over a flat plate of finite thickness with the lower surface held at a constant temperature. An interesting aspect of the solution is that the temperature of the upper surface is found to depend only on Br_x and is independent of Pr.

In the analysis of Luikov [4] two approximate solutions are given. One is based on a differential analysis assuming a uniform velocity profile in the thermal boundary layer and the other is an integral analysis based on polynomial representations of velocity and temperature profiles. As such his results are applicable to Prandtl numbers near or lower than unity. The results of the present study could therefore be considered to complement and extend those of Luikov.

FORMULATION OF THE PROBLEM

Figure 1 shows the geometry and the coordinate system. The problem may be stated mathematically as follows,

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2} \tag{2}$$

$$x = 0 \qquad \theta(0, y) = 0 \tag{3}$$

$$y = 0 \qquad K_f\frac{\partial \theta}{\partial y} = K_s(\theta_\omega - 1) \tag{4}$$

$$y = \infty \qquad \theta(x, \infty) = 0 \tag{5}$$

where θ is the dimensionless temperature defined as,

$$\theta = \frac{T - T_\infty}{T_b - T_\infty}, \quad \theta_\omega = \frac{T_\omega - T_\infty}{T_b - T_\infty} \tag{6}$$

and T_ω is the variable temperature of the upper surface.

Boundary condition (4) is based on the continuity of heat flux and temperature at the solid–fluid interface and the assumption of a linear temperature variation from the lower to the upper surface of the plate.

Under the assumption of large Pr, the thermal boundary layer δ_T is within the hydrodynamic boundary layer δ and

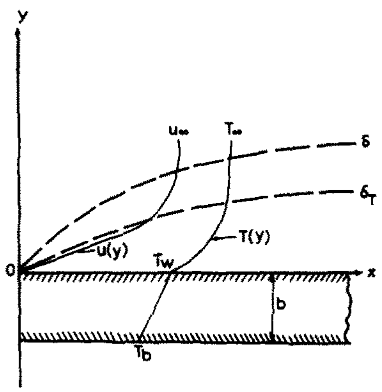

Fig. 1. The geometry and the coordinate system.

therefore a linear velocity variation within δ_T is assumed, i.e.

$$u = \frac{\tau_\omega}{\mu}y = 0.332\frac{u_\infty^{3/2}}{\nu^{1/2}}x^{-1/2}y \tag{7}$$

and from continuity equation,

$$v = 0.083\frac{u_\infty^{3/2}}{\nu^{1/2}}x^{-3/2y^2} \tag{8}$$

substitution of equations (7) and (8) in equation (2) and introduction of new variables ζ and Z defined as,

$$\zeta = \frac{\alpha}{4}\int_0^x \sqrt{\frac{2\tau_\omega}{\mu}}\,\mathrm{d}x \tag{9}$$

$$Z = \sqrt{\frac{\tau_\omega}{2\mu}}\,y \tag{10}$$

transforms equation (2) to the following form,

$$\frac{\partial \theta}{\partial \zeta} = \frac{1}{Z}\frac{\partial^2 \theta}{\partial Z^2}. \tag{11}$$

Taking Laplace transform of both sides of (11) with respect to ζ the following ordinary differential equation is obtained for variable $\bar{\theta}$, the Laplace transform of θ,

$$\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}Z^2} - pZ\bar{\theta} = 0. \tag{12}$$

From the solution of equation (12) subject to the conditions that $\bar{\theta}(0) = \bar{\theta}_\omega$ and $\bar{\theta}(\infty) = 0$, a relation is obtained between $\bar{\theta}_\omega$ and $\bar{Q}_\omega$,

$$\bar{Q}_\omega = \frac{\Gamma(2/3)}{3^{2/3}\Gamma(4/3)}p^{1/3}\bar{\theta}_\omega \tag{13}$$

where Q_ω is a transformed heat flux function defined as,

$$Q_\omega = \frac{q_\omega}{K_f(T_b - T_\infty)}\sqrt{\frac{2\mu}{\tau_\omega}} \tag{14}$$

However, from the assumption of linear temperature variation through the plate, Q_ω is found to be,

$$Q_\omega = 3.78\frac{K_s}{K_f}\frac{Pr^{1/3}}{u_\infty b}(1 - \theta_\omega)\zeta^{1/3}. \tag{15}$$

Elimination of $\bar{Q}_\omega$ from equation (13) using equation (15) results in a single equation for $\bar{\theta}_\omega$,

$$\beta\mathscr{L}[\zeta^{1/3}(1 - \theta_\omega)] = Pr^{1/3}\bar{\theta}_\omega \tag{16}$$

where β is a constant defined as,

$$\beta = \frac{5.18k_s Pr^{1/3}}{u_\infty b k_f}. \tag{17}$$

If one introduces a new variable X,

$$X = \beta^{3/2}\zeta \tag{18}$$

equation (16) may be expressed as an integral equation,

$$\theta_\omega(X) = \frac{\Gamma(4/3)}{\Gamma(5/3)}X^{2/3} - \frac{1}{\Gamma(1/3)}\int_0^X \frac{y^{1/3}\theta_\omega(y)}{(X - y)^{2/3}}\,\mathrm{d}y. \tag{19}$$

For later reference it may be stated that X and Br_x are related as follows,

$$X^{2/3} = \frac{2.175}{Br_x} \tag{20}$$

perhaps it is more convenient to find the wall temperature in terms of $\Phi_\omega(X)$ where $\Phi_\omega(X) = 1 - \theta_\omega(X)$. Making this substitution in (19) results in the integral equation,

$$\Phi_\omega(X) = 1 - \frac{1}{\Gamma(1/3)}\int_0^X \frac{y^{1/3}\Phi_\omega(y)}{(X - y)^{2/3}}\,\mathrm{d}y. \tag{21}$$

The solution of the problem has now been reduced to the solution of the integral equation (21).

It is suitable to present the results in terms of two function N^* and Q^* where N^* is the ratio of Nusselt number in the case that the lower surface is held at a constant temperature to the Nusselt number in the case that upper surface is held at the same constant temperature and Q^* is the ratio of the local heat fluxes in the two cases. It may easily be shown that,

$$N^* = \frac{Nu_x}{Nu_{x0}} = 2.95\frac{\Phi_\omega}{\theta_\omega Br_x} \tag{22}$$

$$Q^* = \frac{q}{q_0} = 2.95\frac{\Phi_\omega}{Br_x}. \tag{23}$$

SOLUTION OF THE PROBLEM

The solution of the integral equation (21) obtained for small, large, and intermediate values of X is given below.

(a) *Small values of* X (*large* Br_x)

This solution is obtained by the method of iteration. An initial assumed $\Phi_w(X)$ is substituted in the integral on the RHS of (21) resulting in a new value for $\Phi_w(X)$ and the procedure is repeated. The first approximation was taken as $\Phi_w(X) = 1.0$. The solution obtained in this way for small X is,

$$\Phi_\omega(X) = 1 + \sum_{i=1}^{\infty} (-1)^i a_i (X^{2/3})^i \tag{24}$$

$$a_i = \frac{\Gamma(4/3)\Gamma(6/3)\ldots\Gamma\left(\dfrac{2i+2}{3}\right)}{\Gamma(5/3)\Gamma(7/3)\ldots\Gamma\left(\dfrac{2i+3}{3}\right)}.$$

(b) *Large values of* X (*small* Br_x)

Noting that as $p \to 0$, $X \to \infty$, $p\bar{\theta}_w \to \theta_w(\infty) = 1$ and expressing equation (19) in the form:

$$\bar{\theta}_\omega = \frac{\Gamma(4/3)}{p^{5/3}} - \frac{1}{p^{1/3}}\mathscr{L}[X^{1/3}\theta_\omega(X)] \tag{25}$$

where the Laplace transform is with respect to X one obtains for large X

$$1 = \frac{\Gamma(4/3)}{p^{2/3}} - p^{2/3}\mathscr{L}[X^{1/3}\theta_\omega(X)]. \tag{26}$$

from which it follows that,

$$\theta_\omega(X) = 1 - \frac{1}{\Gamma(2/3)x^{2/3}}. \tag{27}$$

(c) *Intermediate values of* X

To solve integral equation (21) for intermediate values of X it is first written in equivalent form:

$$\Phi_\omega(X) = 1 - \frac{1}{\Gamma(1/3)}\int_0^X \frac{(X-y)^{1/3}\Phi_\omega(X-y)}{y^{2/3}}\,\mathrm{d}y \tag{28}$$

and by the transformation

$$X = s^3, \quad Y = r^3 \tag{29}$$

the singularity is eliminated,

$$\Phi_\omega(s^3) = 1 - \frac{1}{\Gamma(4/3)}\int_0^s (s^3 - r^3)^{1/3}\Phi_\omega(s^3 - r^3)\,\mathrm{d}r. \tag{30}$$

Finally to put the equation in a form suitable for solution by Gauss, quadrature formula let

$$r = \frac{s}{2}(v+1). \tag{31}$$

In terms of v and X one obtains,

$$\Phi_\omega(X) = 1 - \frac{X^{2/3}}{2\Gamma(4/3)}\int_{-1}^{+1}\left[1 - \left(\frac{v+1}{2}\right)^3\right]^{1/3} \times \Phi_\omega\left\{X\left[1 - \left(\frac{v+1}{2}\right)^3\right]\right\}\mathrm{d}v. \tag{32}$$

The quadrature formula of Gauss with sixteen divisions was used to solve equation (32). Assumed values of Φ_w were substituted in the integral and the integral was evaluated by Gauss's method resulting in new values of $\Phi_w(X)$. The procedure was repeated until new and old values of $\Phi_w(X)$ agreed to within 0.1%. The first estimate for $\Phi_w(X)$ was based on interpolation between the curves for small and large X which were obtained in (a) and (b) above.

RESULTS

Figure 2 shows the variation of $\Phi_w(X)$, N^*, and Q^* with Br_x^{-1}. From the curve for $Q^*(X)$ it is observed that for values of Br_x greater than 0.5 the error in neglecting the thermal resistance of the plate is more than 5% and therefore the problem must be solved as a conjugate one. This is a better criterion than the analysis of N^* because the two Nussult numbers are not based on the same temperature difference.

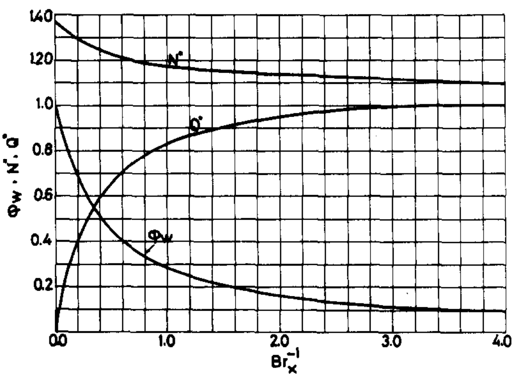

Fig. 2. The graphs of Φ_ω, N^*, and Q^* vs Br_x^{-1}.

However to compare the results with those of Luikov [4] one may obtain from (22) and (27) for large X (small Br_x),

$$N^* \approx 1 + 0.339 Br_x. \tag{33}$$

This is in good agreement with Luikov's integral analysis. Furthermore, it follows from (33) that for Br_x greater than 0.14, N^* will be greater than 1.05. This compares favorably with the value $Br_x = 0.1$ estimated by Luikov.

REFERENCES

1. A. V. Luikov, *Heat and Mass Transfer, Handbook*, Energiya, Moscow (1972).
2. A. V. Luikov, *Analytical Heat Diffusion Theory*. Academic Press, New York (1968).
3. A. V. Luikov, V. A. Alexashenko and A. A. Alexashenko, Analytical methods of solution of conjugated problems in convective heat transfer, *Int. J. Heat Mass Transfer* **14**, 1047 (1971).
4. A. V. Luikov, Conjugate convective heat transfer problems, *Int. J. Heat Mass Transfer* **17**, 257 (1974).
5. M. J. Lighthill, Contributions to the theory of heat transfer through a laminar boundary layer, *Proc. R. Soc.* **A202**, 359 (1950).